STUDY OF GAS MOTION IN A PLANE-PARALLEL CHANNEL WITH CRYOGENIC WALLS

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The Navier–Stokes equations are taken as the basis of a model for describing the motion of a gas in a channel with cryogenic walls. A method for numerical solution of the problem and its results are discussed.

In applied gasdynamics, there is a class of problems that is of interest and importance in applications and involves the motion of a gas in channels and pipes with walls which have cryogenic properties. The temperature of such walls is maintained at a very low level, which ensures freezing of the gas and which, in turn, makes it possible to control flow rate and other dynamic parameters of the gas in a given cross section of the channel by means of purely thermal effects.

In the general case, the description of steady-state flow of a gas in such a plane-parallel channel is provided by the complete Navier–Stokes equations solved in a region whose boundary is determined by the condition of phase transition. A simplified model is proposed below which was constructed without consideration of end effects (infinitely long channel) and under the assumption that the phase-transition surface coincides with the wall at which the appropriate boundary conditions are set. As will be shown in the following, the assumption of a definite law for the variation of viscosity makes it possible to reduce the problem, within the limitations of the model, to the numerical solution of a system of ordinary differential equations.

We consider the steady-state motion of a viscous and thermally conducting gas in an infinitely long, flat channel of width 2R. The gas is assumed thermally and calorically ideal; its Prandtl number and specific heat are constants. The Navier–Stokes equations are written in Cartesian coordinates, where the z axis is directed along the symmetry axis of the channel and the y axis is along the normal to the wall,

$$\frac{\partial(\rho v_y)}{\partial y}+\frac{\partial(\rho v_z)}{\partial z}=0,$$

$$\frac{\partial}{\partial y}(\rho v_y^2)+\frac{\partial}{\partial z}(\rho v_y v_z)+\frac{\partial p}{\partial y}=-\left(\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}\right),$$

$$\frac{\partial}{\partial y}(\rho v_y v_z)+\frac{\partial}{\partial z}(\rho v_z^2)+\frac{\partial p}{\partial z}=-\left(\frac{\partial \tau_{zy}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right), \tag{1}$$

$$\frac{\partial}{\partial y}(\rho v_y H)+\frac{\partial}{\partial z}(\rho v_z H)=-\frac{\partial q_y}{\partial y}-\frac{\partial q_z}{\partial z}-\frac{\partial}{\partial y}(v_y\tau_{yy})$$
$$-\frac{\partial}{\partial z}(v_z\tau_{zz})-\frac{\partial}{\partial y}(v_z\tau_{zy})-\frac{\partial}{\partial z}(v_y\tau_{yz}),\qquad p=\frac{\varkappa-1}{\varkappa}\rho h,$$

where ρ is the density of the gas, $H = h + v^2/2$ is the total enthalpy, and $\varkappa = c_p/c_V$ is the ratio of specific heats. The components of the thermal flux vector and of the viscous stress tensor are given by the expressions

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$$\tau_{yy} = -\mu\left[2\frac{\partial v_y}{\partial y} - \frac{2}{3}\left(\frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\right],$$
$$\tau_{yz} = \tau_{zy} = -\mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right),$$
$$\tau_{zz} = -\mu\left[2\frac{\partial v_z}{\partial z} - \frac{2}{3}\left(\frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\right], \tag{2}$$
$$q_y = -\frac{1}{\sigma}\mu\frac{\partial h}{\partial y}, \qquad q_z = -\frac{1}{\sigma}\mu\frac{\partial h}{\partial z}.$$

We assume an adhesion condition as a boundary condition for the velocity at the wall of the channel. Instead of thermal equilibrium at the boundary of the wall layer (it is the same as the phase-separation boundary), we assume that the temperature reached directly at the surface of the wall goes to zero. Such a simplification is widely accepted in the solution of problems with phase transitions [1, 2].

These conditions are added to the condition for symmetry of the flow and we have in all

$$v_y(R,\ z) = v_z(R,\ z) = h(R,\ z) = 0,$$
$$v_y(0,\ z) = \left(\frac{\partial v_z}{\partial y}\right)_{y=0} = \left(\frac{\partial h}{\partial y}\right)_{y=0} = \left(\frac{\partial \rho}{\partial y}\right)_{y=0} = \left(\frac{\partial p}{\partial y}\right)_{y=0} = 0. \tag{3}$$

Integral conditions should be added to the conditions (3); one should assign mass, momentum, and energy flows in the axial direction for the entry cross section of the channel,

$$\int_0^R \rho v_z dy = Rq_1, \qquad \int_0^R (\rho v_z^2 + p + \tau_{zz})\, dy = Rq_2,$$
$$\int_0^R (\rho v_z H + v_z\tau_{zz} + v_y\tau_{yz} + q_z)\, dy = Rq_3, \tag{4}$$

where q_1, q_2, and q_3 are given positive constants.

For closure of the system (1), it is necessary to assign a law for the variation of viscosity. We set this law in the form

$$\mu = A\sqrt{h} \qquad (A = \text{const}), \tag{5}$$

which makes it possible to pick out a class of solutions with separable variables for the system (1) [3]. Let α be some dimensionless parameter and let the relations

$$\frac{\partial \varphi_i}{\partial z} = -k_i\frac{\alpha}{2R}\varphi_i,$$
$$\varphi_1 = \rho v_z, \qquad \varphi_2 = \rho v_y v_z + \tau_{yz}, \qquad \varphi_3 = \rho v_z^2 + p + \tau_{zz}, \tag{6}$$
$$\varphi_4 = \rho v_z H + v_z\tau_{zz} + v_y\tau_{yz} + q_z,$$
$$k_1 = 1, \quad k_2 = k_3 = 2, \quad k_4 = 3$$

be satisfied for the local flow characteristics.

Integrating all terms of the differential equations (1) with respect to y between the limits 0 and R, with Eqs. (6), the boundary conditions (3), and the integral conditions (4) taken into account, we obtain at the entry cross section

$$q_1 = \frac{2}{\alpha}(\rho v_y)|_0^R = \frac{2}{\alpha}(\rho v_y)_{y=R},$$
$$q_2 = \frac{1}{\alpha}(\rho v_y v_z + \tau_{zy})|_0^R = -\frac{1}{\alpha}\left(\mu\frac{\partial v_z}{\partial y}\right)_{y=R}, \tag{7}$$

$$q_3 = \frac{2}{3\alpha}(\rho v_y H + q_y + v_y\tau_{yy} + v_z\tau_{zy})|_0^R = -\frac{2}{3\sigma\alpha}\left(\mu\,\frac{\partial h}{\partial y}\right)_{y=R}.$$

Thus the use of a class of solutions for which Eqs. (6) are satisfied leads to a situation where the integral conditions (4) take the form of the boundary conditions (7) assigned at y = R for certain combinations of the hydrodynamic parameters and their derivatives.

The integral flow characteristics q_i from Eq. (4), the constant A from Eq. (5), and the channel half-width R form a set of dimensional definitive parameters of the problem. We introduce combinations of them having the dimensionality of enthalpy and velocity by assuming

$$h_* = \left(\frac{9\alpha\sigma q_3 R}{4A}\right)^{2/3}, \qquad v_* = \frac{2q_2}{3\sigma q_3}\,h_*. \tag{8}$$

Because Eqs. (6) are satisfied, Eqs. (1) can be transformed to dimensionless form by means of the substitutions

$$\begin{gathered} y = Ry^0, \qquad z = Rz^0, \qquad v_z = v_* w^0 \exp(-\alpha z^0/2), \\ v_y = (\alpha/2)\,v_* v^0 \exp(-\alpha z^0/2), \qquad h = h_* h_1^0 \exp(-\alpha z^0), \\ \rho = \frac{q_1}{v_*}\,\rho^0, \qquad p = \frac{\varkappa-1}{\varkappa}\,\frac{q_1}{v_*}\,h_* p_1^0 \exp(-\alpha z^0), \\ \mu = A\sqrt{h_*}\sqrt{h_1^0}\exp(-\alpha z^0/2), \end{gathered} \tag{9}$$

where w^0, v^0, h^0_1, ρ^0, and p^0_1 are dimensionless functions of the single argument y^0.

We also introduce the dimensionless parameters

$$\begin{gathered} \beta = \frac{\sqrt{h_*}}{v_*} = \left(\frac{3\sigma q_3}{2q_2 v_*}\right)^{1/2} = \left(\frac{3\sigma^2 q_3^2 A}{2\alpha q_2^3 R}\right)^{1/3}, \\ \gamma = \frac{\alpha q_2}{q_1 v_*} = \left(\frac{2\alpha\sigma q_3 A^2}{3q_1^3 R^2}\right)^{1/3}. \end{gathered} \tag{10}$$

Omitting the zeros from the notation for dimensionless quantities and denoting derivatives with respect to y^0 by primes, we obtain

$$\begin{gathered} (\rho v)' = \rho w, \qquad p_1 = \rho h_1, \qquad \frac{\alpha^2}{4}\,\rho v(v' - w) = -\frac{\varkappa-1}{\varkappa}\,\beta^2 p_1' \\ +\alpha\gamma\left[\sqrt{h_1}\left(v' + \frac{1}{2}\,w\right)\right]' - \frac{3}{2}\,\alpha\gamma\sqrt{h_1}\left(w' - \frac{\alpha^2}{4}\,v\right), \\ \frac{\alpha}{2}\,\rho(vw' - w^2) = \frac{\varkappa-1}{\varkappa}\,\alpha\beta^2 p_1 + \frac{3}{2}\,\gamma\left[\sqrt{h_1}\left(w' - \frac{\alpha^2}{4}\,v\right)\right]' + \alpha^2\gamma\sqrt{h_1}\left(w + \frac{1}{2}\,v'\right), \\ \frac{\alpha}{3}\,\rho(vh_1' - 2wh_1) - \frac{\alpha}{3}\cdot\frac{\varkappa-1}{\varkappa}(vp_1' - 2wp_1) = \frac{\gamma}{\sigma}\,(\sqrt{h_1}h_1')' \\ + \frac{3\alpha^2\gamma}{2\sigma}\,h_1^{3/2} + \frac{\gamma}{\beta^2}\sqrt{h_1}\left[\frac{\alpha^2}{2}\,v'^2 + \frac{\alpha^2}{2}\,w^2 + \left(\frac{\alpha^2}{4}\,v - w'\right)^2 - \frac{\alpha^2}{6}(v' - w)^2\right] \end{gathered} \tag{11}$$

after substitution of Eqs. (8) into Eqs. (1).

The boundary conditions obtained from Eqs. (3) and (7) take the form

$$\begin{gathered} v(1) = w(1) = h_1(1) = 0, \qquad \rho(1)\cdot v(1) = 1, \\ \sqrt{h_1(1)}\,w'(1) = -2/3, \qquad \sqrt{h_1(1)}\,h_1'(1) = -2/3, \\ v(0) = w'(0) = h_1'(0) = \rho'(0) = p_1'(0) = 0. \end{gathered} \tag{12}$$

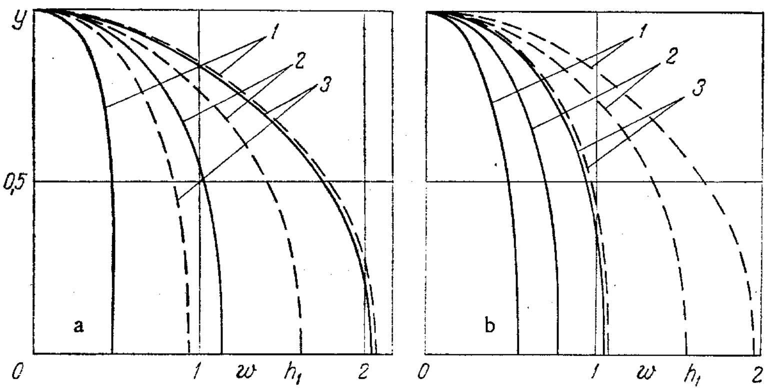

Fig. 1. Profiles of axial velocity and temperature components: a) γ = const = 0.18 [1) β^2 = 1.25; 2) β^2 = 1.45; 3) β^2 = 1.70]; b) β^2 = const = 1.3 [1) γ = 0.185; 2) γ = 0.150; 3) γ = 0.115] (solid curves, velocity; dashed curves, temperature).

Desiring to delineate the nature of the solution of the nonlinear boundary-value problem (11), (12), we write down asymptotic expansions of the functions sought in the neighborhood of the point y = 1 in the form

$$\begin{aligned}
\rho &= \frac{1}{A_v}(1-y)^{-2/3}[1-B_v(1-y)^{2/3}+O(1-y)],\\
v &= A_v(1-y)^{2/3}[1+B_v(1-y)^{2/3}+O(1-y)],\\
w &= (1-y)^{2/3}[1+B_w(1-y)^{2/3}+O(1-y)],\\
h_1 &= (1-y)^{2/3}[1+B_h(1-y)^{2/3}+O(1-y)^{4/3}],\\
p_1 &= \frac{1}{A_v}[1+(B_h-B_v)(1-y)^{2/3}+O(1-y)],
\end{aligned} \tag{13}$$

where the coefficients B_v, B_w, and B_h can be expressed through the assigned parameters and the undetermined coefficient A_v. Assigning some value to the latter, we calculate values for the functions sought (and for their derivatives where necessary) at some initial point $y_0 = 1 - \varepsilon$ (ε is a small quantity). After this, one should integrate Eqs. (11) numerically, varying the value of A_v until all the boundary conditions at y = 0 are satisfied to the required accuracy. Refinement of the value of A_v is accomplished by Newton's method using asymptotic representations of the solution in the neighborhood of y = 0.

The proposed method for numerical solution of the problem may not yield the desired results, since there are physical considerations and data from numerical experiments which furnish evidence that a mathematical solution of the problem (11), (12) does not always exist and that, consequently, the set of parameters α, β, and γ cannot be assigned completely arbitrarily. A rigorous investigation of the boundaries of the region in which a solution exists in (α, β, γ) space presents great mathematical difficulty and is not in accord with our purpose. However, a necessary condition for the existence of a solution can be obtained rather simply if integral relations are obtained from Eqs. (4) by transforming the integrands in them by means of the substitutions (9). Omitting the details of this part of the study, we merely point out that by the substitution into those relations of sufficiently simple but realistic approximations to the desired functions, one can manage not only to check whether a solution of the problem exists for selected values of the dimensionless similarity parameters, but also to select an approximate value of A_v which significantly accelerates and simplifies the numerical integration.

Some results of numerical calculations performed for α = 0.1, $\varkappa$ = 1.4, σ = 0.75, and for a number of values of β^2 and γ are shown in Figs. 1-3. The figures make it particularly clear that an increase in β^2 at fixed γ leads to an elongation of the velocity profile and a compression of the temperature profile, while an increase in γ at fixed β^2 produces the opposite effect.

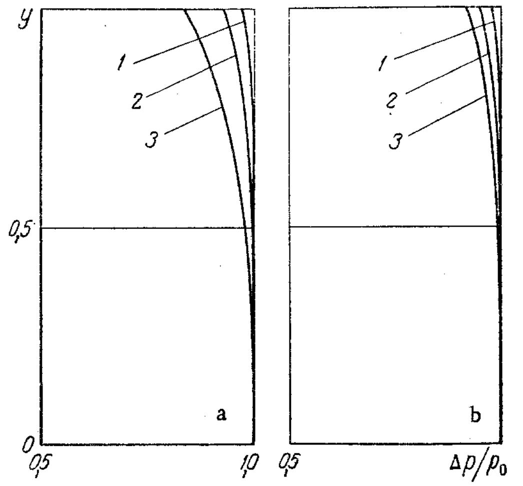

Fig. 2. Pressure profiles: a) γ = const = 0.18 [1) β^2 = 1.25; 2) β^2 = 1.45; 3) β^2 = 1.70]; b) β^2 = const = 1.3 [1) γ = 0.185; 2) γ = 0.150; 3) γ = 0.115].

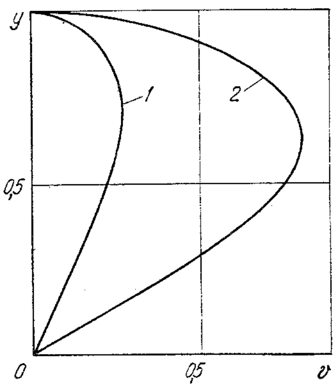

Fig. 3. Profiles of transverse velocity component: 1) γ = 0.185; β^2 = 1.3; 2) γ = 0.170, β^2 = 1.7.

An important characteristic of gasdynamic flow in a pipe or channel is the coefficient of resistance λ_f of the test object. We define it, as is done in gasdynamics [4], by means of the expression (in dimensional notation)

$$\frac{dp_0}{dz} = -\frac{\lambda_f}{2R} \cdot \frac{\rho_0 v_0^2}{2}, \tag{14}$$

where the subscript 0 refers to values on the channel axis at an arbitrary cross section z = const. After transformation to dimensionless notation in accordance with Eqs. (9) and the use of Eqs. (10), we obtain

$$\lambda_f = 4\alpha \frac{\varkappa - 1}{\varkappa} \beta^2 \frac{h_0}{w_0^2} = 2\alpha \frac{\varkappa - 1}{\varkappa} \zeta(\alpha, \beta, \gamma). \tag{15}$$

In the general case, the function $\zeta(\alpha, \beta, \gamma)$ cannot be represented in closed analytic form and yields poor approximations. If the values of α and γ are sufficiently small (we neglect their squares), one can obtain the approximate expression

$$\zeta = B_1 \frac{\varkappa^2}{(\varkappa-1)^2}\left[\frac{3\sigma}{4}\cdot\frac{\gamma}{\alpha\beta^2} - \frac{\varkappa-1}{\varkappa}\right] + \frac{\varkappa}{\varkappa-1}\sqrt{B_1^2 \frac{\varkappa^2}{(\varkappa-1)^2}\left[\left(\frac{3\sigma}{4}\right)^2\left(\frac{\gamma}{\alpha\beta^2}\right)^2 - \frac{3\sigma(\varkappa-1)}{2\varkappa}\cdot\frac{\gamma}{\alpha\beta^2}\right] + \frac{3\sigma}{2} B_2 \frac{\gamma}{\alpha\beta^2}}, \tag{16}$$

where B_1 and B_2 are Euler integrals of the first kind, $B_1 = B(1/2, 5/3)$, and $B_2 = B(1/2, 7/3)$. The requirement that the quantity under the square root sign in Eq. (16) be positive yields yet another limitation on the applicability of Eq. (16) which takes the form $\gamma/(\alpha\beta^2) \geq 0.8535$ when $\varkappa = 1.4$ and $\sigma = 0.75$.

The total resistive force acting on a channel segment of length l is given by

$$F_l = kS[(p_0)_{z=0} - (p_0)_{z=l}] = \lambda_f kS \frac{\rho_0 U^2}{4\alpha}\left(1 - e^{-\alpha\frac{l}{R}}\right), \tag{17}$$

where k is the ratio between the average pressure over a cross section and the pressure at the axis.

The determination of the coefficient of heat transfer of the gas at the channel surface also offers no difficulty in principle, but this coefficient is of little interest in the present instance. We merely point out that, as follows from the physical formulation of the problem, the thermal flux at the wall is a finite, nonzero quantity varying in the axial direction like exp $(-3\alpha z/2)$. Such a nonuniform relase of heat from the channel walls is automatically accomplished in practice with a steady-state flow provided the total cooling is sufficient to ensure wall temperatures close to absolute zero.

This study gives a sufficiently complete representation of the qualitative features of gas flow in a flat channel with cryogenic walls as confirmed by the results of numerical calculations for a typical range of the definitive parameters.

NOTATION

y, z, spatial coordinates; v_y, v_z, components of gas velocity; ρ, density; p, pressure; h, enthalpy; H, total enthalpy; $\varkappa$, ratio of specific heats; τ_{yy}, τ_{yz}, τ_{zz}, components of viscous stress tensor; q_y, q_z, components of thermal flux vector; μ, coefficient of viscosity; σ, Prandtl number; R, channel half-width; q_1, q_2, q_3, flows of mass, momentum, and energy; α, dimensionless parameter characterizing axial flow variations; β, γ, dimensionless flow parameters [Eqs. (10)]; A_v, B_v, B_w, B_h, coefficients of expansions in the neighborhood of the wall; λ_f, coefficient of channel resistance; l, channel length; F_l, total resistive force of channel; S, cross-sectional area of channel; U, velocity at channel axis in entry cross section; 0, subscript denoting values on the axis.

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MATHEMATICAL SIMULATION OF HEAT- AND MASS-TRANSFER PROCESSES IN SEPARATED FLOWS WITH A LAMINAR MIXING REGION AT LOW REYNOLDS NUMBERS

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The lower limit of applicability of the mathematical model of heat and mass transfer constructed by Batchelor and Lavrent'ev for separated flow past a bottom trench is extended.

The mathematical model of heat and mass transfer for separated flow past a bottom trench constructed by Batchelor and Lavrent'ev [1] is of undisputed interest in many chemical-engineering problems. In particular, it is used successfully for study of hydrodynamic inhomogeneities in reactors with a fixed catalyst layer [2].

Quantitative estimates of heat and mass transfer are obtained in this model by employing dynamic and diffusion boundary-layer theory, so that its use is recommended only at Reynolds numbers exceeding hundreds or even thousands [3]. However, a significant number of processes take place at moderate or low Reynolds numbers, also with realization of a separation in the flow, so that this model could also be utilized. Numerical solutions of the Navier–Stokes equation by Myshenkov [4] for flow of a gas beyond a plate of finite thickness in the Reynolds number range of 1.7 to 100 show the possibility of existence of flows with separation at even small Reynolds numbers. The gas flow in the wake beyond the plate at $Re < 1.7$ is of a continuous nature, but at $Re = 1.7$ at the rear critical point there develops

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